Chapter 19

Credit Risk or Default Risk

19.1. Introduction

As mentioned by Basel I and Basel II Committees, the credit risk problem is one of the most important contemporary problems for banks and insurance companies. Financial studies have been developed both from theoretical and practical points of views. They consist of calculating the default probability of a firm.

There is a very wide range of research on credit risk models (see, for example, Bluhm *et al.* (2002), Crouhy *et al.* (2000), Lando (2004), etc.).

In the 1990s, Markov models were introduced to study credit risk problems. Many important papers on these kinds of models were published (see Jarrow and Turnbull (1995), Jarrow *et al.* (1997), Nickell *et al.* (2000), Israel *et al.* (2001), and Hu *et al.* (2002)), mainly for solving the problem of the evaluation of the transition matrices. In Lando and Skodeberg (2002) some problems regarding the duration of the transition are expressed, but never, as far as the authors know, a model in which the randomness of time in the states transitions has been constructed.

Semi-Markov models were introduced by Janssen, Manca and D'Amico (2005a) and Janssen and Manca (2007) firstly in the homogenous case. The non-homogenous case was developed in Janssen, Manca and D'Amico (2004a) and Janssen and Manca (2007). With these new models, it is possible to generalize the Markov models introducing the randomness of time for transitions between the states.

19.2. The Merton model

19.2.1. *Evaluation model of a risky debt*

The *Merton* (1974) model or the *firm model* considers the case of a firm that borrows an amount M of money at time 0, for example in the form of a zero coupon bond with facial value F (interests included) representing the amount to reimburse at time T.

As the borrower has the risk that the firm will be in default at time T, the debt is called a *risky debt* of value $D(0)$ at time 0. This value of the risky debt must use a stochastic model, called here the Merton model.

After the loan, we have:

$$V(0)=A+M, \tag{19.1}$$

$V(0)$ representing the value of the firm at time 0.

At the maturity of debt T, two situations are possible following this value $V(0)$ with respect to F. They are given by the next table.

At time T	$V(T)<F$	$V(T)>F$
Borrowers	$V(T)$	F
Shareholders	0	$V(T)-F$

Table 19.1. *Situation at maturity time*

Using the concept of plain vanilla options, it is clear that the values of $A(T)$ and $D(T)$ representing respectively the *equities of the shareholders* and the *value of the risky debt* are given by:

$$\begin{aligned} A(T) &= \max\{0, V(T)-F\}, \\ D(T) &= \min\{V(T), F\}\,(= F-\max\{0, F-V(T)\}). \end{aligned} \tag{19.2}$$

Thus, at t=0, with the Black and Scholes approach for the evaluation of options, under the risk neutral measure Q and with F as exercise price, we obtain

$$\begin{aligned} &A(0) = e^{-rT} E_Q[\max\{0, V(T) - F\}] \text{ (value of the call)}, \\ &D(0) = Fe^{-rT} - e^{-rT} E_Q[\max\{0, F - V(T)\}] \; (= e^{-rT} F - \text{ put}), \end{aligned} \tag{19.3}$$

r being, as usual, the instantaneous non-risky interest rate.

From this last relation, we obtain:

$$Fe^{-rT} - D(0) = e^{-rT} E_Q[\max\{0, F - V(T)\}], \tag{19.4}$$

which shows that the difference between the non-risky debt and the risky debt is simply the value of the put in the hands of the shareholders taking account of the possibility of default.

Let us recall that Merton uses the traditional Black and Scholes model given in Chapter 14.

So, on the complete filtered space $(\Omega, \Im, (\Im_t), Q)$, the process *value of the firm* $V = (V(t), t \in [0,T])$ satisfies:

$$\begin{aligned} &dV = V(t)rdt + V(t)\sigma dW(t), \\ &V(0) = V_0, \end{aligned} \tag{19.5}$$

and we know that:

$$\begin{aligned} &P(S,t) = Ke^{-r(T-t)}\Phi(-d_2) - S\Phi(-d_1), \\ &d_1 = \frac{1}{\sigma\sqrt{T-t}}\left[\log\frac{S}{K} + (r + \frac{\sigma^2}{2})(T-t)\right], \\ &d_2 = d_1 - \sigma\sqrt{T-t}, \\ &S = S(t). \end{aligned} \tag{19.6}$$

We have:

$$K = F, S = V(0), t = 0,$$

and thus:

$$
\begin{aligned}
&P(V(0),T) = [Fe^{-rT}\Phi(-d_2) - V(0)\Phi(-d_1)],\\
&d_1 = \frac{1}{\sigma\sqrt{T}}\left[\log\frac{V(0)}{Fe^{-rT}} + (r+\frac{\sigma^2}{2})T\right],\\
&d_2 = d_1 - \sigma\sqrt{T}.
\end{aligned}
\tag{19.7}
$$

From relation (19.4), the value of the risky debt is given by:

$$D(0) = Fe^{-rT} - [Fe^{-rT}\Phi(-d_2) - V(0)\Phi(-d_1)],$$

where (19.8)

$$
\begin{aligned}
&d_1 = \frac{1}{\sigma\sqrt{T}}\left[\log\frac{V(0)}{Fe^{-rT}} + (r+\frac{\sigma^2}{2})T\right],\\
&d_2 = d_1 - \sigma\sqrt{T}.
\end{aligned}
$$

19.2.2. *Interpretation of Merton's result*

From relation (19.8), we can write $D(0)$ in the following form:

$$
\begin{aligned}
&D(0) = Fe^{-rT} - \Phi(-d_2)[Fe^{-rT} - V(0)\frac{\Phi(-d_1)}{\Phi(-d_2)}],\\
&d_1 = \frac{1}{\sigma\sqrt{T}}\left[\log\frac{V(0)}{Fe^{-rT}} + (r+\frac{\sigma^2}{2})T\right],\\
&d_2 = d_1 - \sigma\sqrt{T}.
\end{aligned}
\tag{19.9}
$$

The first term is nothing other than the present value at time 0 of the non-risky debt of amount F; the second term is the product of the default probability at time T, $P(V(T) < F)$ and the present value of the expected loss amount

$$\left[Fe^{-rT} - V(0)\frac{\Phi(-d_1)}{\Phi(-d_2)}\right].$$

Let us show for example that $\Phi(-d_2)$ is the *default probability* $(P(V(T) < F)$.

Indeed, from the lognormality property of $V(T)/V(0)$, we successively obtain:

$$\begin{aligned} P(V(T)<F) &= P\left(\frac{V(T)}{V_0}<\frac{F}{V_0}\right) \\ &= P\left(\ln\frac{V(T)}{V_0}<\ln\frac{F}{V_0}\right), \\ &= P\left(\frac{\ln\frac{V(T)}{V_0}-\left(r-\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{t}}<\frac{\ln\frac{F}{V_0}-\left(r-\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{t}}\right), \\ &= \Phi\left(\frac{\ln\frac{F}{V_0}-\left(r-\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{t}}\right). \end{aligned} \tag{19.10}$$

From the Black and Scholes result, we have:

$$\begin{aligned} d_1 &= \frac{\ln\left(\frac{V_0}{K}\right)+\left(r+\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} \\ d_2 &= \frac{\ln\left(\frac{V_0}{K}\right)+(r-\frac{\sigma^2}{2})T}{\sigma\sqrt{T}}(=d_1-\sigma\sqrt{T}), \\ K &= F. \end{aligned} \tag{19.11}$$

So, we obtain the desired result.

19.2.3. *Spreads*

The value of the risky debt $D(0)$ may be seen as the present value of F using a rate r' defined by:

$$D(0)=e^{-r'T}F, \tag{19.12}$$

so that

$$\begin{aligned} &D(0) = e^{-r'T}F, \\ &r' = -\frac{1}{T}\ln\frac{F}{D(0)}. \end{aligned} \tag{19.13}$$

The corresponding spread is thus given by:

$$spread = r' - r. \tag{19.14}$$

To compute the interest rate corresponding to the corresponding non-risky debt, we define the rate r'' such that:

$$M = e^{-r''T}F \tag{19.15}$$

and so:

$$r'' = -\frac{1}{T}\ln\frac{F}{M}. \tag{19.16}$$

This gives another spread as the difference of risky and non-risky rates called actuarial spread:

$$\text{actuarial spread} = r'' - r. \tag{19.17}$$

Example 19.1 (Farber *et al.*, (2004)) A firm has an initial capital of €2,500,000 and for future investments it is necessary to receive a loan of €2,000,000 to be reimbursed in two years.

The firm finds a bank agreeing this loan in the form of a zero coupon bond with facial value €3,000,000, interests included and of course of maturity 2 years.

This gives a rate r'' of 22.5%!

The next table gives the result related to the value of the risky debt.

Data of the firm		
Initial capital A(0)		2,500,000
Facial value F(T)		3,000,000
Volatility		0.6931
Maturity T		2
Amount M		2,000,000
Firm value at t=0:V(0)		4,500,000
Non-risky rate		
Annual		0.02
Instantaneous		0.01980263
Results		
Present value of F		2,883,506.34
d(1)	0.94416045	
d(2)	–0.03603097	
phi(-d(1))	0.172543815	
phi(-d(2))	0.514371227	
Default probability		0.51437123
Current value of recovering		1373998.96
Value of the risky debt: D(0)		2176760.82
Conclusions		
Instantaneous rate of the loan		0.20273255
Annual rate of the loan r''		0.22474487
Instantaneous rate of risky debt		0.16038719
Annual rate of risky debt r'		0.17396533
Spread		0.06435768
Spread with the non-risky rate		
With r''		0.20474487
With r'		0.15396533
Actuarial spread		0.06435768

Table 19.2. *Merton model*

19.3. The Longstaff and Schwartz model (1995)

To improve the Merton model, Longstaff and Schwartz (1995) have introduced a threshold K such that the firm is in default if its value is below K.

To compute the default risk $PDF(T)$ before time T, from the Merton model:

$$\begin{aligned} dV &= \mu V dt + \sigma V dW(t), \\ V(0) &= V_0, \end{aligned} \tag{19.14}$$

we know that

$$\ln\frac{V(t)}{V_0} = \left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W(t). \tag{19.15}$$

It follows that:

$$PDF(T) = P(V(T) < K), \tag{19.16}$$

and so:

$$PDF(T) = P\left(\ln\frac{V(t)}{V_0} < \ln\frac{K}{V_0}\right). \tag{19.17}$$

As from relation (19.15), we obtain:

$$\ln\frac{V(t)}{V_0} \prec N\left(\left(\mu - \frac{\sigma^2}{2}\right)t, \sigma^2 t\right) \tag{19.18}$$

And we obtain from relation (19.17):

$$PDF(T) = \Phi\left(\frac{\ln\frac{K}{V_0} - \left(\mu - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right). \tag{19.19}$$

This is the result of Longstaff and Schwartz (1995) in their model called the KMV Credit Monitor.

It must be clear that this model gives the possibility to be in default at time *t* and no more in default at time *s*, *s*>*t*.

If we introduce, as in Janssen (1993), the concept of lifetime of the firm as the stopping time τ defined as:

$$\tau = \inf\{t:V(t) < K\} \tag{19.20}$$

or as:

$$\tau = \inf\left\{t:\ln\frac{V(t)}{V_0} < \ln\frac{K}{V_0}\right\}. \tag{19.21}$$

With result (19.15), we have:

$$\tau = \inf\left\{t:(\mu - \frac{\sigma^2}{2})t + \sigma W(t) < \ln\frac{K}{V_0}\right\}. \tag{19.22}$$

It follows that:

$$\tau = \inf\left\{t:\ln\frac{V_0}{K} + (\mu - \frac{\sigma^2}{2})t + \sigma W(t) < 0\right\}. \tag{19.23}$$

Finally, with:

$$u = \ln\frac{V_0}{K}, \mu' = \left(\mu - \frac{\sigma^2}{2}\right)t, \tag{19.24}$$

we can write:

$$P(\tau < t) = \Psi(u,t), \tag{19.25}$$

Using the fundamental results of Cox and Miller (1965) on diffusion processes, we finally obtain:

$$\Psi(u,t) = 1 - \Phi\left(\frac{u + \mu' t}{\sigma\sqrt{t}}\right) + e^{-\frac{2\mu' u}{\sigma^2}}\Phi\left(\frac{-u + \mu' t}{\sigma\sqrt{t}}\right). \tag{19.26}$$

This probability is called the ruin probability before *t* in the actuarial *risk theory*, and so the *non-ruin probability before t* is given by:

$$\phi(u,t) = 1 - \Psi(u,t). \tag{19.27}$$

For $t \to \infty$, we obtain:

$$\Psi(u) = \lim_{t\to\infty} \Psi(u,t) = \begin{cases} 1, \mu' \leq 0, \\ e^{-2\frac{\mu'}{\sigma^2}u}, \mu' > 0 \end{cases} \tag{19.28}$$

and so:

$$\phi(u) = \lim_{t\to\infty} \phi(u,t) = \begin{cases} 0, \mu' \leq 0, \\ 1 - e^{-2\frac{\mu'}{\sigma^2}u}, \mu' > 0. \end{cases} \tag{19.29}$$

Remark 19.1 It is clear that the default probability of Longstaff and Schwartz is always smaller than the ruin probability computed by the Janssen model.

19.4. Construction of a rating with Merton's model for the firm

19.4.1. *Rating construction*

In this section, we will develop an elaboration of a rating model using the traditional Merton model for the firm (1974), which is used in Creditmetrics initialized by J.P. Morgan as a sequel of the Riskmetrics computer program dedicated to the VaR methods (see Janssen and Manca (2007)).

In the Merton model (1974), value *V* of the firm is modeled with a Black and Scholes stochastic differential equation with trend μ and instantaneous volatility σ (see Chapter 14)

$$\begin{aligned} dV &= V(t)\mu dt + V(t)\sigma dW(t), \\ V(0) &= V_0, \end{aligned} \tag{19.30}$$

so that its value time at *t* is given by

$$V(t) = V_0 e^{(\mu - \frac{\sigma^2}{2})t + \sigma W(t)} \tag{19.31}$$

V_0 being the value of the firm at time 0 and $W = \left(W(t), t \in [0,T]\right)$ a standard Brownian motion defined on the filtered probability space $\left(\Omega, \Im, (\Im_t), P\right)$.

If V_{def} is the threshold beyond which the firm defaults, called the threshold default, the probability P_{def} that the company defaults before time *t* is given by:

$$\begin{aligned} P_{def}\left(V_{def}, t\right) &= P\left(V(t) < V_{def}\right) \\ &= P\left(\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W(t) < \ln \frac{V_{def}}{V_0}\right) \\ &= P\left(W(t) < \frac{1}{\sigma}\left(\ln \frac{V_{def}}{V_0} - \left(\mu - \frac{\sigma^2}{2}\right)t\right)\right). \end{aligned} \tag{19.32}$$

As, for all positive *t*, $W(t)/\sqrt{t}$ has a normal distribution, we obtain:

$$P_{def}\left(V_{def}, t\right) = \Phi\left(\frac{1}{\sigma\sqrt{t}}\left(\ln \frac{V_{def}}{V_0} - \left(\mu - \frac{\sigma^2}{2}\right)t\right)\right). \tag{19.33}$$

So, if we fix value V_{def}, we can compute the corresponding value of P_{def} using the quartiles of the normal distribution.

Of course, the inverse is possible: first fix P_{def} and then compute the corresponding level V_{def}.

In the following, let us suppose that we fix the default probability V_{def} so that we compute the corresponding quantile Z_{CCC} given by

$$\begin{aligned} &P_{def}\left(V_{def}, t\right) = \Phi\left(Z_{CCC}\right), \\ &Z_{CCC} = \frac{1}{\sigma\sqrt{t}}\left(\ln \frac{V_{def}}{V_0} - \left(\mu - \frac{\sigma^2}{2}\right)t\right). \end{aligned} \tag{19.34}$$

This means that if *Z* is below or equal to Z_{CCC}, with *Z* defined by:

$$Z = \frac{1}{\sigma\sqrt{t}}\left(\ln \frac{V}{V_0} - \left(\mu - \frac{\sigma^2}{2}\right)t\right), \tag{19.35}$$

the considered firm is supposed to be in default and theoretically has to stop all activities.

On the contrary, if the value of Z is larger than Z_{CCC}, corresponding to the threshold value V_{CCC}, but before the quartile Z_B, corresponding to the threshold value V_B, the rating given to the firm is noted *CCC*, etc. So, with a fixed scale of firm threshold values:

$$V_{def} = V_{CCC} < V_B < V_{BB} < V_{BBB} < V_A < V_{AA} < V_{AAA} \tag{19.36}$$

we obtain a scale of increasing thresholds quartiles represented by:

$$Z_{CCC} < Z_B < Z_{BB} < Z_{BBB} < Z_A < Z_{AA} < Z_{AAA}, \tag{19.37}$$

assigning a credit rating or grade to firms as an estimate of their creditworthiness.

If Z represents the observed value of Z for the considered firm, the scale used here is the rating used by the famous credit rating agencies Standard and Poor's, and Moody's given below.

Zobs value	**notation**
Zobs<ZCCC	default
ZCCC<Zobs<ZB	CCC
ZB<Zobs<ZBB	B
ZBB<Zobs<ZBBB	BB
ZBBB<Zobs<ZA	BBB
ZA<Zobs<ZAA	A
ZAA<Zobs<ZAAA	AA
ZAAA<Zobs	AAA

Table 19.3. *Rating agencies*

It is clear that the credit ratings depend on time t and also on the selection of the probabilities

$$P_{def} \quad P(Z_{CCC}),\ P(Z_B),\ P(Z_{BB}),\ P(Z_{BBB}),\ P(Z_A),\ P(Z_{AA}),\ P(Z_{AAA}) \tag{19.38}$$

or on the threshold scale of firm values

$$Z_{CCC} < Z_B < Z_{BB} < Z_{BBB} < Z_A < Z_{AA} < Z_{AAA} \tag{19.39}$$

chosen by the credit rating agency.

We can also compute the following relations:

$$
\begin{aligned}
&P_{déf} = P(Z_{obs} < Z_{CCC}),\\
&P_{CCC} = P(Z_{CCC} < Z_{obs} < Z_{B}),\\
&P_{B} = P(Z_{B} < Z_{obs} < Z_{BB}),\\
&P_{BB} = P(Z_{BB} < Z_{obs} < Z_{BBB}),\\
&P_{BBB} = P(Z_{BBB} < Z_{obs} < Z_{A}),\\
&P_{A} = P(Z_{A} < Z_{obs} < Z_{AA}),\\
&P_{AA} = P(Z_{AA} < Z_{obs} < Z_{AAA}),\\
&P_{AAA} = P(Z_{AAA} < Z_{obs}),
\end{aligned}
\tag{19.40}
$$

and so:

$$
\begin{aligned}
&P_{B} = P_{def} + P_{CCC,}\\
&P_{def} + P_{CCC} + P_{B} + \cdots + P_{AA} + P_{AAA} = 1.
\end{aligned}
\tag{19.41}
$$

Using relation (19.35), we obtain:

$$
\begin{aligned}
&P_{def} = \Phi\left(\frac{1}{\sigma\sqrt{t}}\left(\ln\frac{V_{CCC}}{V_0} - \left(\mu - \frac{\sigma^2}{2}\right)t\right)\right),\\
&P_{def} + P_{CCC} = \Phi\left(\frac{1}{\sigma\sqrt{t}}\left(\ln\frac{V_{B}}{V_0} - \left(\mu - \frac{\sigma^2}{2}\right)t\right)\right),\\
&P_{def} + P_{CCC} + P_{B} = \Phi\left(\frac{1}{\sigma\sqrt{t}}\left(\ln\frac{V_{BB}}{V_0} - \left(\mu - \frac{\sigma^2}{2}\right)t\right)\right),\\
&\ldots\\
&P_{def} + P_{CCC} + P_{B} + P_{BB} + \cdots + P_{AA} = \Phi\left(\frac{1}{\sigma\sqrt{t}}\left(\ln\frac{V_{AAA}}{V_0} - \left(\mu - \frac{\sigma^2}{2}\right)t\right)\right);
\end{aligned}
\tag{19.42}
$$

and moreover:

$$\begin{aligned}
P_{déf} &= \Phi\left(\frac{1}{\sigma\sqrt{t}}\left(\ln\frac{V_{CCC}}{V_0}-\left(\mu-\frac{\sigma^2}{2}\right)t\right)\right),\\
P_{CCC} &= \Phi\left(\frac{1}{\sigma\sqrt{t}}\left(\ln V\frac{Z_B}{V_0}-\left(\mu-\frac{\sigma^2}{2}\right)t\right)\right)-\Phi\left(\frac{1}{\sigma\sqrt{t}}\left(\ln\frac{V_{CCC}}{V_0}-\left(\mu-\frac{\sigma^2}{2}\right)t\right)\right),\\
P_B &= \Phi\left(\frac{1}{\sigma\sqrt{t}}\left(\ln\frac{V_{BB}}{V_0}-\left(\mu-\frac{\sigma^2}{2}\right)t\right)\right)-\Phi\left(\frac{1}{\sigma\sqrt{t}}\left(\ln\frac{V_B}{V_0}-\left(\mu-\frac{\sigma^2}{2}\right)t\right)\right),\\
&\dots\\
P_{AA} &= \Phi\left(\frac{1}{\sigma\sqrt{t}}\left(\ln\frac{V_{AAA}}{V_0}-\left(\mu-\frac{\sigma^2}{2}\right)t\right)\right)-\Phi\left(\frac{1}{\sigma\sqrt{t}}\left(\ln\frac{V_{AA}}{V_0}-\left(\mu-\frac{\sigma^2}{2}\right)t\right)\right),\\
P_{AAA} &= 1-\Phi\left(\frac{1}{\sigma\sqrt{t}}\left(\ln\frac{V_{AAA}}{V_0}-\left(\mu-\frac{\sigma^2}{2}\right)t\right)\right).
\end{aligned} \tag{19.43}$$

All these relations show how the grades are time dependent, which is why we will now study the dynamics of ratings.

19.4.2. *Time dynamic evolution of a rating*

19.4.2.1. *Continuous time model*

In continuous time, the rating process is nothing other than the stochastic process defined by relation (19.33),

$$Z = \left\{Z_t, 0 \le t \le T\right\} \tag{19.44}$$

where r.v. Z_t represents the credit rating at time t given by:

$$P_{def}(V_t, t) = \Phi(Z_t),$$

or (19.45)

$$Z_t = \frac{1}{\sigma\sqrt{t}}\left(\ln\frac{V_t}{V_0}-\left(\mu-\frac{\sigma^2}{2}\right)t\right).$$

Here, grade Z_t represents exactly the value inside one of the classes defined above and no longer only the class.

Substituting the value of V_t from relation (19.30) in (19.45), we obtain:

$$Z_t = \frac{W(t)}{\sqrt{t}}, t > 0, \tag{19.46}$$

so that

$$P\left(Z_{t+\Delta t} \le j \middle| Z_t\right) = P\left(\frac{W(t+\Delta t)}{\sqrt{t+\Delta t}} \le j \middle| \frac{W(t)}{\sqrt{t}} = i\right), \Delta t > 0, i, j > Z_{CCC}. \tag{19.47}$$

As the standard Brownian process has stationary and independent increments (see Definition 10.27), we also obtain:

$$\begin{aligned} &P\left(\frac{W(t+\Delta t)}{\sqrt{t+\Delta t}} \le j \middle| \frac{W(t)}{\sqrt{t}} = i\right) \\ &= P\left(W(t+\Delta t) - W(t) \le j\sqrt{t+\Delta t} - W(t) \middle| \frac{W(t)}{\sqrt{t}} = i\right), \end{aligned} \tag{19.48}$$

or using relation (19.47):

$$\begin{aligned} P(Z_{t+\Delta t} \le j | Z_t) &= P\left(\frac{W(t+\Delta t) - W(t)}{\sqrt{\Delta t}} \le \frac{j\sqrt{t+\Delta t} - i\sqrt{t}}{\sqrt{\Delta t}} \middle| Z_t = i\right) \\ &= \Phi\left(\frac{j\sqrt{t+\Delta t} - i\sqrt{t}}{\sqrt{\Delta t}}\right), \end{aligned} \tag{19.49}$$

the last equality coming from the normality of the increments of a standard Brownian motion.

We can also write this last result in the form:

$$P(Z_s \le j | Z_t = i) = \Phi\left(\frac{j\sqrt{s} - i\sqrt{t}}{\sqrt{s-t}}\right). \tag{19.50}$$

The corresponding density function is given by:

$$\frac{d}{dj}\Big(P(Z_s \le j | Z_t = i)\Big) = \frac{\sqrt{s}}{\sqrt{s-t}} \Phi'\left(\frac{j\sqrt{s} - i\sqrt{t}}{\sqrt{s-t}}\right). \tag{19.51}$$

This last result is correct only for $i \geq Z_{CCC}$. On the other hand, for $i < Z_{CCC}$, the default state being considered as an absorbing state, we have necessarily for $j \geq i$:

$$P(Z_s \leq j \mid Z_t = i) = 1. \tag{19.52}$$

In conclusion, as the transition probability given by (4.21) depends on both *s* and *t* and not only on *t* – *s*, we proved that the *Z* process is *a non-homogenous Markov process*, introduced in Chapter 3.

19.4.2.2. *Discrete time model*

Let us define $\{1,...,m\}$ as the set of the *m* credit ratings ranked in increasing order with Moody's scale: 1= D_{def} (*default*), 2=Z_{CCC},..., *m*=Z_{AAA}.

Except for the extreme classes, the rating classes defined below will now be represented by their centers as follows:

$$\begin{aligned}
&(-\infty,1] &:\quad& 1\\
&(1,2] &:\quad& \frac{3}{2}\\
&....\\
&(i-1,i] &:\quad& \frac{2i-1}{2}\\
&...\\
&(m-1,m] &:\quad& \frac{2m-1}{2}\\
&(m,\infty) &:\quad& m
\end{aligned} \tag{19.53}$$

Let $Z_t = i$, *i* being a class center different from 1; from result (19.50), we have:

$$\begin{aligned}
&P(j-1 < Z_s \leq j \mid Z_t = i)\\
&= \Phi\left(\frac{j\sqrt{s} - i\sqrt{t}}{\sqrt{s-t}}\right) - \Phi\left(\frac{(j-1)\sqrt{s} - i\sqrt{t}}{\sqrt{s-t}}\right), \qquad s > t.
\end{aligned} \tag{19.54}$$

To obtain a discrete-time, let us suppose that we give notations at times 0,*u*,2*u*,…,*ku* representing for example one year or a semester. Now transition probabilities become:

$$\begin{aligned}
&P(j-1 < Z_{ku+1} \leq j \mid Z_{ku} = i)\\
&= \Phi\left(\frac{j\sqrt{ku+1} - i\sqrt{ku}}{\sqrt{u}}\right) - \Phi\left(\frac{(j-1)\sqrt{ku+1} - i\sqrt{ku}}{\sqrt{u}}\right), k = 0,1,.....
\end{aligned} \tag{19.55}$$

Of course, if Z_{ku} equals Z_{Def} , we know from relation (19.52) that

$$P(j-1<Z_{ku+1}\leq j\,|\,Z_{ku}=Z_D)=\begin{cases}0, & j>1,\\ 1, & j\leq 1.\end{cases}\tag{19.56}$$

Relations (19.54) and (19.55) define a sequence of probability transition matrices $\mathbf{P}(k)$, k=0,1,... with:

$$\mathbf{P}(k)=\left[\,p_{ij}(k)\,\right]\tag{19.57}$$

and

$$p_{ij}(k)=P(j-1<Z_{ku+1}\leq j\,|\,Z_{ku}=i),i,\,j=1,...,m,k=0,1,...\tag{19.58}$$

It follows that the credit rating process Z in discrete-time Z=(Z_{ku},k=0,1,...) is what we call a non-homogenous Markov chain defined in Chapter 12.

Of course, in the very particular and unrealistic case where the probability transition matrices $\mathbf{P}(k)$, k=0,1,... are independent of t, the process in discrete-time Z=(Z_{ku},k=0,1,...,) is then a homogenous Markov chain as defined in Chapter 11.

19.4.2.3. *Example*

In real-life economics, credit rating agencies play a crucial role; they compile data on individual companies or countries to estimate their probability of default, represented by their scale of credit ratings at a given time and also by the probability of transitions for successive credit ratings.

A change in the rating is called a *migration*. Migration to a higher rating will of course increase the value of a company’s bond and decrease its yield, giving what we call a negative *spread*, as it has a lower probability of default, and the inverse is true with a migration towards a lower grade with consequently a positive spread.

Here we have an example of a possible transition matrix for migration from one year to the next one.

	AAA	**AA**	**A**	**BBB**	**BB**	**B**	**CCC**	**D**	**Total**
AAA	0.90829	0.08272	0.00736	0.00065	0.00066	0.00014	0.00006	0.00012	1
AA	0.00665	0.9089	0.07692	0.00583	0.00064	0.00066	0.00029	0.00011	1
A	0.00092	0.0242	0.91305	0.05228	0.00678	0.00227	0.00009	0.00041	1
BBB	0.00042	0.0032	0.05878	0.87459	0.04964	0.01078	0.0011	0.00149	1
BB	0.00039	0.00126	0.00644	0.0771	0.81159	0.08397	0.0097	0.00955	1
B	0.00044	0.00211	0.00361	0.00718	0.07961	0.80767	0.04992	0.04946	1
CCC	0.00127	0.00122	0.00423	0.01195	0.0269	0.11711	0.64479	0.19253	1
D	0	0	0	0	0	0	0	1	1

Table 19.4. *Example of transition matrix of credit ratings*

We clearly see that the probabilities of no migration, given by the elements of the principal diagonal, are the highest elements of the matrix but that they decrease with the poor quality of the rating.

Here, we see for example that a company with rank AA has more or less nine chances out of 10 to keep its rating next year but it will move to rank AAA with only six chances in 1,000.

On the other hand, a company with a CCC as a rating will be in default next year with 20 chances out of 100.

As a more concrete example, the next table gives the transition probability matrix of *Standard and Poor's* credit ratings for 1998 (see ratings performance, Standard and Poor's) for a sample of 4,014 companies.

Let us point out the presence of a "new" state called NR (*rating withdrawn*) meaning that for a company in such a state, the rating has been withdrawn and that this event does not necessary lead to default the following year, thus explaining the last row of the above matrix.

Effec.		**AAA**	**AA**	**A**	**BBB**	**BB**	**B**	**CCC**	**D**	**NR**	**Total**
165	**AAA**	90	6	0	0.61	0	0	0	0	3.03	100
560	**AA**	0.18	89.8	5.61	0.18	0	0	0	0	4.23	100
1,095	**A**	0.09	1.5	87.18	5.11	0.18	0	0	0	5.94	100
896	**BBB**	0	0	2.79	84.93	4.46	0.67	0.22	0.34	6.59	100
619	**BB**	0.32	0.2	0.16	5.33	75.4	5.98	2.75	0.65	9.21	100
649	**B**	0	0	0.15	0.62	6.16	76.27	5.09	4.47	7.24	100
30	**CCC**	0	0	3.33	0	0	20	33.31	36.69	6.67	100
	NR	0	0	0	0	0	0	0	0	100	100
4,014											

Table 19.5. *Example with rating withdrawn*

Here, we see for example that companies in state AA will not be in default the next year but that 5.61% of them will degrade to A and 0.18% to a BBB and 0.18% will upgrade to an AAA.

Under the assumption of a homogenous Markov chain, we obtain the following results:

(i) *the probability that an AA company defaults after two years*

$$P^{(2)}(D/AA)=0.0018\cdot 0.0034=0.0006\%,$$

which is still very low;

(ii) *the probability that a BBB company defaults in one of the next two years*

This probability is given by:

$$\begin{aligned} P(D/BBB;2) &= P(D/BBB)+P(BBB/BBB)P(D/BBB) \\ &+P(BB/BBB)P(D/BB)+P(B/BBB)P(D/B)+P(CCC/BBB)P(D/CCC) \\ &= 0.34\%+(84.93\%\cdot 0.34\%)+(4.46\%\cdot 0.65\%)+(0.67\%\cdot 4.47\%)+(0.22\%\cdot 36.67\%) \\ &=0.77\%; \end{aligned}$$

(iii) *the probability for a company BBB to default between year 1 and year 2*

Using the standard definition of conditional probability (see Chapter 1) we obtain

$$\begin{aligned} P(D\ at\ 2/non\text{-}def.\ at\ 1) &= \mathrm{P}(D\ at\ 2\ and\ non\text{-}def.\ at\ 1)/P(non\text{-}def.\ at\ 1) \\ &=(0.77\%-0.34\%)/(1-0.34\%) \\ &=0.43\%. \end{aligned}$$

Let us point out that these illustrative results are true under the homogenous Markov chain model and moreover give similar results for all the companies of the panel in the same credit rating.

In fact, in real life applications, credit rating agencies also study each company on its own account so that specific information is also determined for giving the final grade.

19.4.2.4. *Ratings and spreads on zero bonds*

Let us first recall that a zero coupon bond is a contract paying a known fixed amount called the *principal*, at some given future date, called the *maturity date*.

So, if the principal is one monetary unit and T the maturity date, the value of this zero coupon at time 0 is given by:

$$B(0,T)=e^{-\delta T} \tag{19.59}$$

if δ is the considered constant instantaneous intensity of interest rate.

Of course, the investor in zero coupons must take into account the risk of default of the issuer. To do so, we consider that, in a risk neutral framework, the investor has no preference between the following two investments:

(i) to receive almost surely at time 1 the amount e^{δ} as counterpart of the investment at time 0 of one monetary unit;

(ii) to receive at time 1 the amount $e^{(\delta+s)}(s>0)$ with probability $(1-p)$ or 0 with probability p, as counterpart of the investment at time 0 of one monetary unit, p being the default probability of the issuer.

The positive quantity s is called the *spread* with respect to the non-risky instantaneous interest rate δ as counterpart of this risky investment in zero coupon bonds.

From the indifference given above, we obtain the following relation:

$$e^{\delta}=(1-p)e^{(\delta+s)} \tag{19.60}$$

or

$$1=(1-p)e^{s}, \tag{19.61}$$

$$s=-\ln(1-p). \tag{19.62}$$

$$s\approx p,$$

$$s\cong p+\frac{1}{2}p^{2}. \tag{19.63}$$

Let us now consider a more positive and realistic situation in which the investor can obtain an amount $\alpha,(0<\alpha<1)$ if the issuer defaults at maturity or before.

In this case, the expectation equivalence principle relation (19.60) becomes:

$$e^{\delta}=(1-p)e^{\delta+s}+p\alpha e^{\delta}, \tag{19.64}$$

or

$$1=(1-p)e^{s}+p\alpha. \tag{19.65}$$

It follows that in this case the value of the spread satisfies the equation

$$e^s = \frac{1 - p\alpha}{1 - p} \tag{19.66}$$

and so the spread value is

$$s = \ln \frac{1 - p\alpha}{1 - p}. \tag{19.67}$$

As above, using the MacLaurin formula respectively of order 1 and 2, we obtain the two following approximations for the spread:

$$\begin{aligned} s &\approx \frac{p}{1-p}(1-\alpha), \\ s &\approx \frac{p}{1-p}(1-\alpha) - \frac{1}{2}\left(\frac{p}{1-p}(1-\alpha)\right)^2. \end{aligned} \tag{19.68}$$

19.5. Discrete time semi-Markov processes

19.5.1. *Purpose*

In this section, we will present both discrete-time homogenous (DTHSMP) and non-homogenous (DTNHSMP) semi-Markov processes and how to apply semi-Markov models to the credit risk environment.

Although, in general, time in real-life problems is continuous, the real observation of the considered system is almost always made up of discrete-time even if the used time unit may in some cases be very small.

The choice of this time unit depends on what we observe and what we wish to study.

For example, if we are studying the random evolution of the earthquake activity in a tectonic fracture zone, then it could be observed with a unitary time scale of ten years. If we are studying the behavior of a disablement resulting from a job related illness, the unitary time could be one year, etc.

Thus, it results that while the phenomenon of time evolution is continuous, usually, the observations are discrete in time.

Consequently, if we construct a model to be fitted with real data, in our opinion, it would be better to begin with discrete-time models.

The rating changes can be followed by a Markov chain model.

In some papers, the problem of the unfitting of Markov process in the credit risk environment was outlined (see Altman (1998), Nickell *et al.* (2000), Kavvathas (2001), Lando and Skodeberg (2002)).

The principal problems of non-Markovianity that are highlighted are as follows:

(i) the duration inside a state. The probability of changing rating depends on the time that a firm remains at the same rating;

(ii) the dependence on time of the rating evaluation (ageing phenomenon). This means that, in general, the rating evaluation depends on the time at which it is done and, more importantly, on the business cycle. The rating evaluation done at time t is generally different from the one done at time s, if $s \neq t$;

(iii) the dependence of the new rating on the previous ones, not only the last rating, but also the one before last.

As the first approach, the first problem can be solved by means of semi-Markov processes (SMP). In fact, in SMP the transition probabilities are a function of the waiting time spent in a state of the system. Furthermore, in a semi-Markov backward recurrence time conditioning the problem is resolved successfully.

As a general approach, the second problem can be faced by means of a non-homogenous environment and, using a more particular approach, by means of different scenarios in the model.

The third effect exists in the case of downward moving ratings but not in the case of upward moving ratings; see Kavvathas (2001). More precisely, if a firm obtains a lower rating, then there is a higher probability that the next rating will be lower than the preceding one. In the case of an upward movement, this phenomenon does not hold.

The credit risk semi-Markov approach was developed in D'Amico *et al.* (2005a), D'Amico *et al.* (2004a), D'Amico *et al.* (2004b) and D'Amico *et al.* (2005b). In the last sections of this chapter we will present the models and their theoretical background.

It should be mentioned that Koopman *et al.* (2005) and Vasileiou and Vassiliou (2006), in other environments, show how semi-Markov processes are more suitable than the Markov ones in the credit risk transition models.

19.5.2. *DTSMP definition*

Though DTHSMP and DTNHSMP definitions are similar to the continuous ones given in Chapter 3, we will give these definitions for discrete-time using directly the terminology used for continuous time models.

Let I={1, 2, …, m} be the state space and let $\{\Omega, \Im, P\}$ be a probability space. Let us also define the following r.vs.:

$$J_n : \Omega \to I, \qquad T_n : \Omega \to \mathbb{N}. \tag{19.69}$$

Definition 19.1 *The process* (J_n, T_n) *is a discrete-time homogenous Markov renewal process or a discrete-time non-homogenous Markov renewal process if the kernels* **Q** *associated with the process are defined respectively in the following way*:

$$\mathbf{Q} = [Q_{ij}(t)] = [P(J_{n+1} = j,\ T_{n+1} - T_n \le t \mid J_n = i)] \quad i, j \in I, t \in \mathbb{N}, \tag{19.70}$$

$$\mathbf{Q} = [Q_{ij}(s,t)] = [P(J_{n+1} = j,\ T_{n+1} \le t \mid J_n = i, T_n = s)]\ i, j \in I,\ s,t \in \mathbb{N} \tag{19.71}$$

As in the continuous time case, it results that for the homogenous case, we define:

$$\mathbf{P} = [p_{ij}] = \left[\lim_{t\to\infty} Q_{ij}(t)\right];\ i, j \in I,\ t \in \mathbb{N}. \tag{19.72}$$

For the non-homogenous case, we obtain:

$$\mathbf{P} = [p_{ij}(s)] = \left[\lim_{t\to\infty} Q_{ij}(s,t)\right];\ i, j \in I,\ s,t \in \mathbb{N}, \tag{19.73}$$

P being the transition matrix of the *embedded Markov chain* of the process.

Furthermore it is necessary to introduce the probability that the process will leave state i before or at time t:

$$\mathbf{H} = [H_i(t)] = [P(T_{n+1} - T_n \le t \mid J_n = i)], \tag{19.74}$$

$$\mathbf{H} = [H_i(s,t)] = [P(T_{n+1} \le t \mid J_n = i, T_n = s)]. \tag{19.75}$$

From the results of Chapter 12, we know that:

$$H_i(t) = \sum_{j=1}^{m} Q_{ij}(t) \text{ and } H_i(s,t) = \sum_{j=1}^{m} Q_{ij}(s,t). \tag{19.76}$$

Probability (19.77) only has sense in the discrete-time case and to be concise, we present first the definition for the homogenous case and then for the non-homogenous case.

Definition 19.2 *Matrix* **B** *is defined as follows:*

$$\mathbf{B} = \left[b_{ij}(t) \right] = \left[P\left(J_{n+1} = j,\ T_{n+1} - T_n = t \mid J_n = i \right) \right], \tag{19.77}$$

$$\mathbf{B} = \left[b_{ij}(s,t) \right] = \left[P\left(J_{n+1} = j,\ T_{n+1} = t \mid J_n = i, T_n = s \right) \right]. \tag{19.78}$$

From Definition 19.1 it results that:

$$b_{ij}(t) = \begin{cases} Q_{ij}(0) = 0 & \text{if} \quad t = 0, \\ Q_{ij}(t) - Q_{ij}(t-1) & \text{if} \quad t = 1,2,..., \end{cases} \tag{19.79}$$

$$b_{ij}(s,t) = \begin{cases} Q_{ij}(s,s) = 0 & \text{if} \quad t \le s, \\ Q_{ij}(s,t) - Q_{ij}(s,t-1) & \text{if} \quad t > s. \end{cases} \tag{19.80}$$

Definition 19.3 *The discrete-time conditional distribution functions of the waiting times given the present and the next states, are given by:*

$$\mathbf{F} = \left[F_{ij}(t) \right] = \left[P\left(T_{n+1} - T_n \le t \mid J_n = i, J_{n+1} = j \right) \right], \tag{19.81}$$

$$\mathbf{F} = \left[F_{ij}(s,t) \right] = \left[P\left(T_{n+1} \le t \mid J_n = i, J_{n+1} = j, T_n = s \right) \right]. \tag{19.82}$$

Obviously, the related probabilities can be obtained by means of the following formulae:

$$F_{ij}(t) = \begin{cases} Q_{ij}(t) / p_{ij} & \text{if} \quad p_{ij} \ne 0, \\ U_1(t) & \text{if} \quad p_{ij} = 0, \end{cases} \tag{19.83}$$

$$F_{ij}(s,t) = \begin{cases} Q_{ij}(s,t) / p_{ij}(s) & \text{if} \quad p_{ij}(s) \ne 0, \\ U_1(s,t) & \text{if} \quad p_{ij}(s) = 0, \end{cases} \tag{19.84}$$

where $U_1(t) = U_1(s,t) = 1\ \forall s,t$.

Now, we can introduce the *discrete-time semi-Markov process* $Z = \left(Z(t), t \in \mathbb{N} \right)$ where $Z(t) = J_{N(t)}$, $N(t) = \max\left\{ n : T_n \le t \right\}$ represents the state occupied by the process at time *t*.

For $i,j=1,\ldots,m$, the transition probabilities are defined in the following way:

$$\phi_{ij}(t) = \mathrm{P}\left(Z_t = j \mid Z_0 = i \right) \tag{19.85}$$

for the homogenous case; for the non-homogenous case, we have:

$$\phi_{ij}(s,t) = \mathrm{P}\left(Z_t = j \mid Z_s = i, N(s^+) = N(s^-) + 1 \right). \tag{19.86}$$

They are obtained by solving the following evolution equations:

$$\phi_{ij}(t) \;=\; \delta_{ij}(1 - H_i(t)) + \sum_{\beta=1}^{m}\sum_{\vartheta=1}^{t} b_{i\beta}(\vartheta)\phi_{\beta j}(t-\vartheta), \tag{19.87}$$

$$\phi_{ij}(s,t) \;=\; \delta_{ij}(1 - H_i(s,t)) + \sum_{\beta=1}^{m}\sum_{\vartheta=s+1}^{t} b_{i\beta}(s,\vartheta)\phi_{\beta j}(\vartheta,t), \tag{19.88}$$

where, as usual, δ_{ij} represents the Kronecker symbol.

The first part of relations (19.87) and (19.88)

$$\delta_{ij}(1 - H_i(t)) \tag{19.89}$$

$$\delta_{ij}(1 - H_i(s,t)) \tag{19.90}$$

give the probability that the system does not have transitions up to time t given that it was in state i at time 0 in the homogenous case and at time s in the non-homogenous case. Relations (19.89) and (19.90) in the rating migration case represent the probability that the rating organization does not give any new rating evaluation in a time t in homogenous case and from the time s up to the time t in non-homogenous case. This part has sense if and only if $i=j$ and this is the reason of Kronecker δ.

In the second parts

$$\sum_{\beta=1}^{m}\sum_{\vartheta=1}^{t} b_{i\beta}(\vartheta)\phi_{\beta j}(t-\vartheta)$$

$$\sum_{\beta=1}^{m}\sum_{\vartheta=s+1}^{t} b_{i\beta}(s,\vartheta)\phi_{\beta j}(\vartheta,t) \tag{19.91}$$

$b_{i\beta}(\vartheta)$ and $b_{i\beta}(s,\vartheta)$ represent the probability that the system was at time s in the state I, remained in this state up to time ϑ and that it went to the state β just at time ϑ. After the transition, the system will go to state *j* following one of the possible trajectories that go from state β at time ϑ to state *j* within time t. In the credit risk environment, this means that in a time *t*, in the homogenous case, and from time *s* up to time ϑ, in the non-homogenous, the rating company does not offer any other evaluation of the firm; at time ϑ the rating company gave the new rating β for the evaluation firm. After this, the rating will arrive at state *j* within the time *t* following one of the possible rating trajectories.

19.6. Semi-Markov credit risk models

The rating process, generated by the rating agency, gives a reliability rating to a firm's bond.

For example, in Standard and Poor's case, there are the eight different classes of rating which means having the following set of states:

$$I = \{\text{AAA, AA, A, BBB, BB, B, CCC, D}\}.$$

The first seven states are good states and the last one is the only bad state that is also the only absorbing state. The two subsets are the following:

$$U = \{\text{AAA, AA, A, BBB, BB, B, CCC}\}, \quad D = \{\text{ D}\}.$$

Solving systems (19.88) and (19.89) we will obtain the following results:

1) $\phi_{ij}(t)$ and $\phi_{ij}(s,t)$ represent the probabilities of being in state *j* starting in state *i* after time *t* in the homogenous case, or starting at time *s* in state *i* in the non-homogenous one. Both the results take into account the different probabilities of changing state during the permanence of the system in the same state (duration problem). In the non-homogenous case, the problem of the different probabilities of changing state as a function of the different time of evaluation (aging problem) is also solved.

2) $A_i(t) = \sum_{j \in U} \phi_{ij}(t)$ and $A_i(s,t) = \sum_{j \in U} \phi_{ij}(s,t)$ represent the probability that the system never goes in the default state in time *t* in homogenous case and from time *s* up to the time *t* in the non-homogenous one.

3) $1 - H_i(t)$ and $1 - H_i(s,t)$ represent the probability that in time *t* or from time *s* up to the time *t*, no new rating evaluation was done for the firm.

Before giving another result that can be obtained in an SMP environment, we have to introduce the concept of the first transition after time *t*. More precisely, we suppose that the system at time 0 or at time *s* was in state *i*, and we know that with probability $\left(1-H_i(t)\right)$ or $\left(1-H_i(s,t)\right)$ the system does not move from state *i*. According to these hypotheses we would know the probability that the next transition will be to state *j*. This probability will be denoted by $\varphi_{ij}(t)$ in the homogenous case and by $\varphi_{ij}(s,t)$ in the non-homogenous case. These probabilities have the following meaning:

$$\varphi_{ij}(t) = P\left[\, X_{n+1} = j \mid X_n = i, T_{n+1} - T_n > t \,\right] \tag{19.92}$$

$$\varphi_{ij}(s,t) = P\left[\, X_{n+1} = j \mid X_n = i, T_{n+1} > t, T_n = s \,\right]. \tag{19.93}$$

These probabilities can be obtained by means of the following relations:

$$\varphi_{ij}(t) = \frac{p_{ij} - Q_{ij}(t)}{1-H_i(t)} \tag{19.94}$$

$$\varphi_{ij}(s,t) = \frac{p_{ij}(s) - Q_{ij}(s,t)}{1-H_i(s,t)}. \tag{19.95}$$

After definitions (19.92) and (19.93) by means of SMP, it is possible to obtain the following results:

4) $\varphi_{ij}(t)$ and $\varphi_{ij}(s,t)$ represent, respectively, the probabilities of obtaining rank *j* at the next rating if the previous state was *i* and no rating evaluation was done in time *t* in the homogenous case, or from time *s* up to time *t* in the non-homogenous one. In this way, for example, if the transition to the default state is possible and if the system does not move from time *s* up to time *t* from state *i*, we know the probability that in the next transition the system will go to the default state.

The downward problem can be solved introducing six other states. The set of the states becomes the following:

$$I = \left\{\text{AAA, AA,AA-, A,A-, BBB,BBB-, BB,BB-, B,B-, CCC,CCC-, D}\right\}$$

For example, state BBB is divided into BBB and BBB-. The system will be in state BBB if it arrived from a lower rating. On the other hand, it will be in state BBB- if it arrived in the state from a better rating (a downward transition).

It is also possible to suppose that if there is a virtual transition, then if the system is in the BBB- state it will go to the BBB state, but in our models this assumption will not be made.

The first 13 states are good states and the last one is the only bad state. According to this hypothesis, the two subsets become the following:

$$U = \{\text{AAA, AA,AA-, A,A-, BBB,BBB-, BB,BB-, B,B-, CCC,CCC-}\},$$
$$D = \{\ \text{D}\}$$
.

The homogenous and non-homogenous models do not change. The simple introduction of the states makes it possible to solve the downward problem.

19.7. NHSMP with backward conditioning time

Now we introduce non-homogenous backward semi-Markov process, that is, a generalization of the SMP. We state only the non-homogenous case. To explain the backward introduction in Figure 7.1 a trajectory of a SMP with backward recurrence time is shown.

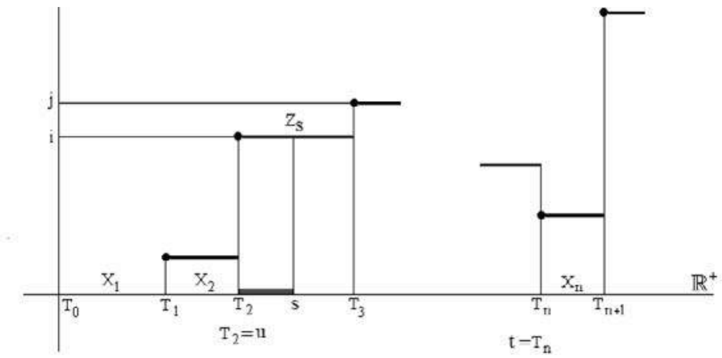

Figure 19.1. *Backward time conditioning*

With non-homogenous semi-Markov processes we know that at time *s* the system entered into state *I*, then the probability of being in state *j* at time *t* is given by $\phi_{ij}(s,t)$. Taking into account backward time, we consider that we entered into the state *i* at time *u*, and we remained in state *i* up to time *s* (backward recurrence time *s*-*u*). The transition probabilities are conditioned to the entrance time into state *i* and to the fact that the system does not have transitions up to time *s*. So, we introduce the new conditional probabilities

$$H_i(u,s,t) = \mathrm{P}\left[T_{N(s)+1} \le t \mid J_{N(s)} = i, T_{N(s)} = u, T_{N(s)+1} > s\right], t > s,$$
$$Q_{ij}(u,s,t) = \mathrm{P}\left[T_{N(s)+1} \le t, J_{N(s)+1} = j \mid J_{N(s)} = i, T_{N(s)} = u, T_{N(s)+1} > s\right], t > s$$

It is clear that $H_i(s,s,t)$, $\dot{Q}_i(s,s,t)$ and $\phi_{ij}(s,s,t)$ are equal respectively to $H_i(s,t)$, $\dot{Q}_i(s,t)$ and $\phi_{ij}(s,t)$ of the non-homogenous semi-Markov process.

According to this hypothesis, relations (19.71), (19.91), (19.88) and (19.90) are rewritten in the following way:

$$Q_{ij}(u,s,t) = \frac{Q_{ij}(u,u,t)}{1-H_i(u,u,s)} \tag{19.96}$$

$$b_{ij}(u,s,t) = \frac{b_{ij}(u,u,t)}{1-H_i(u,u,s)} \tag{19.97}$$

$$\phi_{ij}(u,s,t) = D_{ij}(u,s,t) + \sum_{j\in I}\sum_{\vartheta=s+1}^{t} \phi_{\beta j}(\vartheta,\vartheta,t) b_{i\beta}(u,s,\vartheta) \tag{19.98}$$

where

$$D_{ij}(u,s,t) = \begin{cases} \dfrac{1-H_i(u,u,t)}{1-H_i(u,u,s)} & \text{if} \quad i = j \\ 0 & \text{if} \quad i \neq j \end{cases} \tag{19.99}$$

and $\phi_{ij}(u,s,t)$ is the probability of being in state j at time t given that at time s the system was in state i and that it entered into this state at time u and has not moved from state i up to time s.

With this generalization of the model it is possible to consider the complete time of duration into a state in the rating migration model.

The results given in the previous section with backward conditioning recurrence time become the following:

1) $\phi_{ij}(u,s,t)$ represents the probability of being in state j at time t being in state i at time s and moreover given that the system arrived at state i at time u and that from u to s $(u < s)$ there was no transition. These results take into account the different probabilities of changing state during the permanence of the system in the same state (duration problem) considering the arrival time in the state and, in a complete way, the duration inside a state. Furthermore, it also considers the different probabilities of changing state as a function of the different time of evaluation (aging problem). The different probability values given for the two states that are obtained because of the downward problem solve the third Markovian model problem.

2) $A_i(u,s,t)=\sum_{j\in U}\phi_{ij}(u,s,t)$ represents the probability that the system never goes in the default state from time *s* up to time *t*.

3) $D_{ii}(u,s,t)$ represents the probability that from time *s* up to time *t* no one new rating evaluation was done for the firm, taking into account that there were no transitions from *u* to *s* either.

In this case, $\varphi_{ij}(s,t)$ does not make sense because the backward gives no more information as regards the case without recurrence times.

19.8. Examples

In this section, we present examples for the homogenous case and for downward and backward non-homogenous models; for a simple non-homogenous case see Janssen and Manca (2007). The data were extracted from Standard and Poor's Credit Review (1993), and Standard and Poor's (2001).

19.8.1. *Homogenous SMP application*

The first example is given using the transition matrix given in Jarrow *et al.* (1997), who presented one the first applications of Markov processes to the problem of credit risk.

Real data were not available and this matrix is used only in order to show how the model can work and the results that can be obtained by means of a homogenous semi-Markov process model.

The matrix was constructed starting from the 1 year transition matrix given in Standard and Poor's Credit Review (1993). The matrix is given in Table 19.6 for the sake of completeness.

The d.f. of waiting times are not known and they were constructed by means of random number generators.

The results at 5 years and at 10 years of the matrix are reported $\phi_{ij}(t)$ respectively in Tables 19.7 and 19.8.

	AAA	**AA**	**A**	**BBB**	**BB**	**B**	**CCC**	**D**
AAA	0.891	0.0963	0.0078	0.0019	0.003	0	0	0
AA	0.0086	0.901	0.0747	0.0099	0.0029	0.0029	0	0
A	0.0009	0.0291	0.8896	0.0649	0.0101	0.0045	0	0.0009
BBB	0.0006	0.0043	0.0656	0.8428	0.0644	0.016	0.0018	0.0045
BB	0.0004	0.0022	0.0079	0.0719	0.7765	0.1043	0.0127	0.0241
B	0	0.0019	0.0031	0.0066	0.0517	0.8247	0.0435	0.0685
CCC	0	0	0.0116	0.0116	0.0203	0.0754	0.6492	0.2319
D	0	0	0	0	0	0	0	1

Table 19.6. *1 year transition matrix*

For example, element 0.03046 in row **A** and in column **BBB** represents the probability that a firm that at time 0 has a rating **A** will have rating **BBB** at time 5.

	AAA	**AA**	**A**	**BBB**	**BB**	**B**	**CCC**	**D**
AAA	0.94730	0.04462	0.00474	0.00142	0.00185	0.00007	0.00000	0.00001
AA	0.00437	0.93638	0.04961	0.00616	0.00166	0.00176	0.00002	0.00005
A	0.00049	0.01130	0.94901	0.03046	0.00516	0.00289	0.00005	0.00065
BBB	0.00036	0.00232	0.03778	0.91369	0.03290	0.00886	0.00140	0.00268
BB	0.00027	0.00123	0.00366	0.04166	0.89871	0.03611	0.00472	0.01363
B	0.00000	0.00102	0.00219	0.00577	0.02916	0.90182	0.01727	0.04277
CCC	0.00000	0.00004	0.00570	0.00497	0.00718	0.02673	0.86863	0.08675
D	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	1.00000

Table 19.7. *Probabilities* $\phi_{ij}(5)$

	AAA	**AA**	**A**	**BBB**	**BB**	**B**	**CCC**	**D**
AAA	0.83816	0.13758	0.01566	0.00376	0.00437	0.00038	0.00003	0.00007
AA	0.01099	0.85822	0.10525	0.01633	0.00460	0.00422	0.00012	0.00029
A	0.00133	0.03697	0.85606	0.08135	0.01528	0.00688	0.00026	0.00185
BBB	0.00087	0.00628	0.08211	0.79992	0.07740	0.02256	0.00292	0.00794
BB	0.00051	0.00300	0.01241	0.08615	0.73333	0.11574	0.01495	0.03391
B	0.00003	0.00282	0.00533	0.01253	0.06824	0.75073	0.05572	0.10460
CCC	0.00001	0.00027	0.01325	0.01395	0.02199	0.08238	0.61142	0.25673
D	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	1.00000

Table 19.8. *Probabilities* $\phi_{ij}(10)$

In Table 19.9, the $A_i(t)$ values are reported, the probabilities of not having a default in a time *t* (row index) starting in state *i* (column) at time 0.

	AAA	**AA**	**A**	**BBB**	**BB**	**B**	**CCC**	**D**
1	1.00000	1.00000	0.99995	0.99946	0.99826	0.98987	0.98989	0.0
2	1.00000	1.00000	0.99982	0.99902	0.99379	0.98636	0.98264	0.0
3	1.00000	0.99999	0.99969	0.99848	0.99296	0.98072	0.95688	0.0
4	0.99999	0.99997	0.99954	0.99825	0.99081	0.97239	0.93484	0.0
5	0.99999	0.99995	0.99935	0.99732	0.98637	0.95723	0.91325	0.0
6	0.99998	0.99993	0.99924	0.99633	0.98303	0.95100	0.86029	0.0
7	0.99998	0.99989	0.99900	0.99560	0.97817	0.93408	0.82584	0.0
8	0.99997	0.99984	0.99882	0.99444	0.97353	0.91576	0.77271	0.0
9	0.99995	0.99978	0.99850	0.99327	0.96946	0.90660	0.76244	0.0
10	0.99993	0.99971	0.99815	0.99206	0.96609	0.89540	0.74327	0.0

Table 19.9. *Probabilities of not having a default*

	AAA	**AA**	**A**	**BBB**	**BB**	**B**	**CCC**	**D**
1	0.85082	0.92221	0.90662	0.90109	0.92033	0.85765	0.93533	1.0
2	0.72879	0.85294	0.78671	0.81736	0.82019	0.65152	0.90909	1.0
3	0.69140	0.77216	0.67244	0.78712	0.79283	0.61430	0.85917	1.0
4	0.63930	0.65477	0.62791	0.62841	0.72874	0.58226	0.73706	1.0
5	0.47396	0.50142	0.58289	0.56618	0.68413	0.54727	0.61716	1.0
6	0.32902	0.37689	0.41751	0.51725	0.60283	0.32242	0.54618	1.0
7	0.28210	0.32079	0.39316	0.40741	0.47414	0.27700	0.45527	1.0
8	0.12558	0.24453	0.36959	0.25555	0.33608	0.21594	0.32597	1.0
9	0.11273	0.15467	0.19339	0.15823	0.16723	0.20158	0.16877	1.0
10	0.08805	0.02465	0.00905	0.04343	0.02941	0.03959	0.04901	1.0

Table 19.10. *Probability of remaining in the starting state*

As explained before, these results can assume great relevance in the computation of interest rates.

In Table 19.10, the probabilities of remaining in the starting state without transitions are reported.

In Tables 19.11 and 19.12, the probability $\varphi_{ij}(t)$ at 5 years and 10 years are reported. For example, 0.06644 represents the probability that the next transition of a firm that was at time 0 in the state **A** and that remained in this state up to time 5 will go to state **BBB** in the next transition.

	AAA	AA	A	BBB	BB	B	CCC	D
AAA	0.85843	0.12677	0.00993	0.00182	0.00305	0.00000	0.00000	0.00000
AA	0.00949	0.91387	0.06126	0.00951	0.00303	0.00283	0.00000	0.00000
A	0.00084	0.03299	0.88487	0.06644	0.01045	0.00369	0.00000	0.00073
BBB	0.00051	0.00417	0.05568	0.85789	0.06149	0.01481	0.00113	0.00432
BB	0.00024	0.00166	0.00735	0.05012	0.80635	0.10368	0.01260	0.01801
B	0.00000	0.00194	0.00278	0.00436	0.05011	0.82294	0.05565	0.06223
CCC	0.00000	0.00000	0.01027	0.01145	0.02199	0.08057	0.63240	0.24333
D	0.85843	0.12677	0.00993	0.00182	0.00305	0.00000	0.00000	0.00000

Table 19.11. *Probability* $\varphi_{ij}(5)$

	AAA	AA	A	BBB	BB	B	CCC	D
AAA	0.98206	0.00684	0.00766	0.00164	0.00179	0.00000	0.00000	0.00000
AA	0.03037	0.86667	0.07300	0.01369	0.00639	0.00988	0.00000	0.00000
A	0.00342	0.14461	0.15548	0.64345	0.01673	0.03479	0.00000	0.00153
BBB	0.00025	0.00631	0.10388	0.76402	0.09709	0.02065	0.00147	0.00634
BB	0.00107	0.00242	0.00308	0.06613	0.63752	0.17163	0.04287	0.07528
B	0.00000	0.00016	0.00765	0.00036	0.09770	0.69738	0.04657	0.15017
CCC	0.00000	0.00000	0.01742	0.01122	0.03786	0.02115	0.50411	0.40823
D	0.98206	0.00684	0.00766	0.00164	0.00179	0.00000	0.00000	0.00000

Table 19.12. *Probability* $\varphi_{ij}(10)$

As was mentioned before, by means of this matrix it is possible, for example, to know the probability of going into the default state at the next transition.

Finally, Tables 19.13 and 19.14 present the discrete-time distribution functions of the first time of default in a time horizon of 10 years.

	1	2	3	4	5
AAA	0.00000	0.000000	0.000002	0.000005	0.000010
AA	0.00000	0.000004	0.000014	0.000029	0.000047
A	0.00005	0.000181	0.000311	0.000462	0.000648
BBB	0.00054	0.000980	0.001521	0.001749	0.002678
BB	0.00174	0.006214	0.007042	0.009187	0.013634
B	0.01013	0.013635	0.019281	0.027613	0.042766
CCC	0.01011	0.017358	0.043122	0.065157	0.086749
D	1.00000	1.00000	1.00000	1.00000	0.000010

Table 19.13. *Distribution function from 1 to 5*

	6	7	8	9	10
AAA	0.000015	0.000024	0.000033	0.000046	0.000066
AA	0.000074	0.000112	0.000158	0.000218	0.000290
A	0.000756	0.001005	0.001180	0.001503	0.001854
BBB	0.003667	0.004402	0.005563	0.006727	0.007937
BB	0.016971	0.021832	0.026472	0.030544	0.033914
B	0.049002	0.065918	0.084239	0.093395	0.104603
CCC	0.139715	0.174160	0.227294	0.237560	0.256734
D	1.00000	1.00000	1.00000	1.00000	1.00000

Table 19.14. *Distribution function from 6 to 10*

19.8.2. *Non-homogenous downward example*

To solve the downward problem we constructed the non-homogenous embedded Markov chain using the transition matrices given in Standard and Poor's (2001) Table 15 as a basis. In these matrices, the state No Rating was present. Each element $p_{ij}(s)$ of the embedded non-homogenous Markov chain should be constructed directly from the data. Constructing the MC, all the possible transitions from state *i* to state *j* starting from year *s* should be taken into account. Since we do not have the raw data, we used the one year transition matrices given in Standard and Poor's publication.

The publication reports a 20-year history (one year transition matrices from 1981 to 2000). The example works from year 0, corresponding to 1981 to year 19 that corresponds to year 2000. The $\mathbf{P}(s)$ in the semi-Markov environment should give the transition probabilities that there are, theoretically, from time *s* up to ∞. This fact means that if there is a transition from i to j at time $t, s < t$ then $p_{ij}(k) > 0,\ s \le k \le t$. Standard and Poor's transition matrix was rearranged taking into account this property. Furthermore, we rearranged the obtained matrix giving the transition probabilities of the downward states starting from the probability transitions constructed without the added states.

In the new states, the transition probabilities of remaining in the state or of obtaining a better rating are lower than those of the corresponding non-downward state.

	AAA	AA	AA-	A	A-	BBB	BBB-
AAA	0.906284	0	0.074012	0	0.016665	0	0.003039
AA	0.019456	0.890148	0	0	0.068095	0	0.004009
AA-	0.016895	0	0.851902	0	0.085766	0	0.016418
A	0.006028	0.04704	0	0.87366	0	0	0.06546
A-	0.00435	0.040023	0	0	0.818887	0	0.102706
BBB	0.00324	0.006481	0	0.04782	0	0.886292	0
BBB-	0.002331	0.005244	0	0.03467	0	0	0.847431
BB	0	0.005712	0	0.008785	0	0.044019	0
BB-	0	0.005106	0	0.0077	0	0.035284	0
B	0	0.001342	0	0.011884	0	0.006518	0
B-	0	0.001242	0	0.009932	0	0.004489	0
CCC	0.012308	0	0	0.010443	0	0.011375	0
CCC-	0.00027	0	0	0.007712	0	0.007495	0
D	0	0	0	0	0	0	0

Table 19.15. *Embedded MC at time 0 - **I***

	BB	BB-	B	B-	CCC	CCC-	D
AAA	0	0	0	0	0	0	0
AA	0	0.008809	0	0.007235	0	0.002249	0
AA-	0	0.009641	0	0.013364	0	0.006014	0
A	0	0.00208	0	0.00168	0	0	0.004052
A-	0	0.005128	0	0.003674	0	0	0.025232
BBB	0	0.04782	0	0.003044	0	0.002258	0.003044
BBB-	0	0.089055	0	0.004953	0	0.007478	0.008838
BB	0.598413	0	0	0.294862	0	0.004392	0.043817
BB-	0	0.566994	0	0.316054	0	0.019595	0.049267
B	0.047345	0	0.875595	0	0	0.023673	0.033643
B-	0.042402	0	0	0.846713	0	0.056063	0.03916
CCC	0.011096	0	0.084755	0	0.847646	0	0.022378
CCC-	0.010123	0	0.064831	0	0	0.707889	0.20168
D	0	0	0	0	0	0	1

Table 19.16. *Embedded MC at time 0 - **II***

The probabilities of obtaining a lower rating are higher compared to that of the original state.

In Tables 19.15, 19.16, 19.17 and 19.18, two years of the non-homogenous embedded MC are reported.

	AAA	AA	AA-	A	A-	BBB	BBB-
AAA	0.899545	0	0.095701	0	0.004754	0	0
AA	0	0.918598	0	0	0.081402	0	0
AA-	0	0	0.87046	0	0.12954	0	0
A	0.00172	0.005059	0	0.919263	0	0	0.070619
A-	0.001502	0.004106	0	0	0.872126	0	0.100619
BBB	0	0.008356	0	0.052747	0	0.858366	0
BBB-	0	0.007356	0	0.043275	0	0	0.815366
BB	0	0	0	0	0	0.080374	0
BB-	0	0	0	0	0	0.072374	0
B	0	0.003848	0	0	0	0.003848	0
B-	0	0.003102	0	0	0	0.003483	0
CCC	0	0	0	0	0	0.018525	0
CCC-	0	0	0	0	0	0.014452	0
D	0	0	0	0	0	0	0

Table 19.17. *Embedded M.C. at time 10 -* ***I***

	BB	BB-	B	B-	CCC	CCC-	D
AAA	0	0	0	0	0	0	0
AA	0	0	0	0	0	0	0
AA-	0	0	0	0	0	0	0
A	0	0.003339	0	0	0	0	0
A-	0	0.021647	0	0	0	0	0
BBB	0	0.061103	0	0.008356	0	0.005536	0.005536
BBB-	0	0.101103	0	0.010356	0	0.006536	0.016008
BB	0.799118	0	0	0.075855	0	0.017861	0.026791
BB-	0	0.754912	0	0.104586	0	0.027861	0.040267
B	0.061352	0	0.747004	0	0	0.034524	0.149423
B-	0.052135	0	0	0.7002	0	0.124524	0.116555
CCC	0.03705	0	0.074099	0	0.518468	0	0.351858
CCC-	0.034205	0	0.06741	0	0	0.483847	0.400086
D	0	0	0	0	0	0	1

Table 19.18. *Embedded M.C. at time 10 -* ***II***

To apply the model, it is also necessary to construct the d.f. of the waiting time in each state *i*, given that the state successively occupied is known. We do not have data and we constructed them by means of random number generators.

In Tables 19.19 and 19.20, the probabilities $1-H_i(s,t)$ of remaining in the state from *s* to *t* without any transition are given.

Probabilities of no transition from year *s* to year *t*								
years		**AAA**	**AA**	**AA-**	**A**	**A-**	**BBB**	**BBB-**
0	1	0.872181	0.803709	0.86605	0.884755	0.859781	0.950899	0.925139
0	2	0.77428	0.740025	0.84935	0.81325	0.807756	0.92007	0.774197
0	3	0.699214	0.719164	0.68683	0.662296	0.792684	0.840488	0.721149
0	4	0.579474	0.607234	0.604108	0.562696	0.748248	0.677993	0.712246
0	5	0.496353	0.431912	0.44989	0.503661	0.676747	0.644425	0.652798
0	6	0.375035	0.370249	0.33209	0.466088	0.624805	0.594064	0.477243
0	7	0.302027	0.324532	0.266083	0.338748	0.505539	0.411709	0.314356
0	8	0.206926	0.270204	0.179743	0.223977	0.354685	0.229692	0.292188
0	9	0.12002	0.204777	0.139748	0.116117	0.17136	0.115246	0.121753
0	10	0.048179	0.031125	0.056915	0.081423	0.036674	0.083032	0.07514
6	7	0.814415	0.783328	0.899957	0.795045	0.764473	0.731597	0.814112
6	8	0.514639	0.456258	0.78702	0.638951	0.570292	0.4426	0.573669
6	9	0.312577	0.307947	0.187436	0.245597	0.266901	0.299585	0.281691
6	10	0.042352	0.026528	0.085198	0.058273	0.037473	0.011313	0.04723

Table 19.19. *Probabilities* $1-H_i(s,t)$

Probabilities of no transition from year *s* to year *t*							
years		**BB**	**BB-**	**B**	**B-**	**CCC**	**CCC-**
0	1	0.985533	0.894593	0.83837	0.914242	0.883844	0.955493
0	2	0.947469	0.848314	0.693836	0.743081	0.725237	0.840868
0	3	0.858208	0.729037	0.618079	0.713725	0.682691	0.684999
0	4	0.701267	0.636971	0.575733	0.681344	0.578327	0.56305
0	5	0.577156	0.585617	0.537296	0.660163	0.502152	0.521922
0	6	0.463875	0.471031	0.401942	0.492227	0.480255	0.439965
0	7	0.359584	0.351512	0.302279	0.454226	0.344989	0.288244
0	8	0.242471	0.212772	0.241288	0.292127	0.193908	0.219999
0	9	0.136879	0.152736	0.185382	0.114899	0.14218	0.108098
0	10	0.064375	0.043551	0.011903	0.084761	0.043851	0.088229
6	7	0.644256	0.61234	0.682245	0.778307	0.692634	0.662672
6	8	0.539284	0.505809	0.639596	0.460034	0.38039	0.473847
6	9	0.265781	0.178203	0.260959	0.188567	0.158466	0.315536
6	10	0.055535	0.078714	0.04885	0.072713	0.083376	0.021339

Table 19.20. *Probabilities* $1-H_i(s,t)$

In Tables 19.21, 19.22, 19.23 and 19.24 the probabilities $\varphi_{ij}(s,t)$ are reported. These values give the probability that the next transition from the state *i* will be to the state *j* given that there was no transition from the time *s* to the time *t*.

For example, element 0.014583 gives the probability that next transition from rating **AA-** will be to rating **AAA** given that from time 0 up to time 4 there will be no *real* or *virtual* transitions.

	AAA	**AA**	**AA-**	**A**	**A-**	**BBB**	**BBB-**
AAA	0.899421	0	0.080348	0	0.017233	0	0.002998
AA	0.019706	0.879427	0	0	0.078952	0	0.003367
AA-	0.014583	0	0.837132	0	0.099825	0	0.01853
A	0.008036	0.060021	0	0.833406	0	0	0.088538
A-	0.0039	0.033986	0	0	0.839978	0	0.093764
BBB	0.002525	0.005556	0	0.045356	0	0.886689	0
BBB-	0.002212	0.004901	0	0.030534	0	0	0.865008
BB	0	0.00567	0	0.010089	0	0.028391	0
BB-	0	0.003461	0	0.007602	0	0.026574	0
B	0	0.000756	0	0.011137	0	0.007193	0
B-	0	0.001141	0	0.01029	0	0.003632	0
CCC	0.010151	0	0	0.00688	0	0.01011	0
CCC-	0.000246	0	0	0.00715	0	0.006269	0
D	0	0	0	0	0	0	0

Table 19.21. *Probabilities of remaining in state i from years 0 to 4 and after to go in j-**I***

	BB	**BB-**	**B**	**B-**	**CCC**	**CCC-**	**D**
AAA	0	0	0	0	0	0	0
AA	0	0.00794	0	0.008128	0	0.00248	0
AA-	0	0.010174	0	0.015311	0	0.004445	0
A	0	0.002865	0	0.001739	0	0	0.005395
A-	0	0.004857	0	0.002694	0	0	0.020821
BBB	0	0.051625	0	0.003346	0	0.001876	0.003027
BBB-	0	0.07824	0	0.004769	0	0.006562	0.007773
BB	0.677291	0	0	0.232917	0	0.004475	0.041167
BB-	0	0.587472	0	0.309078	0	0.019533	0.04628
B	0.047536	0	0.87162	0	0	0.025383	0.036375
B-	0.028513	0	0	0.863333	0	0.049553	0.043538
CCC	0.009922	0	0.079238	0	0.86236	0	0.021339
CCC-	0.009463	0	0.083273	0	0	0.671109	0.22249
D	0	0	0	0	0	0	1

Table 19.22. *Probabilities of remaining in state i from years 0 to 4 and after to go in j-**II***

	AAA	**AA**	**AA-**	**A**	**A-**	**BBB**	**BBB-**
AAA	0.904587	0	0.086438	0	0.005801	0	0.003173
AA	0.003638	0.935816	0	0	0.026268	0	0.008966
AA-	0.001835	0	0.875386	0	0.059341	0	0.030207
A	0.006464	0.044629	0	0.909804	0	0	0.030872
A-	0.004883	0.053664	0	0	0.852282	0	0.074755
BBB	0.002969	0.007921	0	0.066346	0	0.855137	0
BBB-	0.002141	0.005289	0	0.019131	0	0	0.814275
BB	0	0.005274	0	0.007533	0	0.029585	0
BB-	0	0.003526	0	0.010214	0	0.018422	0
B	0	0.001895	0	0.007703	0	0.005509	0
B-	0	0.000877	0	0.005706	0	0.00593	0
CCC	0.023288	0	0	0.021232	0	0.018959	0
CCC-	0.0005	0	0	0.007729	0	0.007913	0
D	0	0	0	0	0	0	0

Table 19.23. *Probabilities of remaining in state i from years 2 to 7 and after to go in j-**I***

	BB	**BB-**	**B**	**B-**	**CCC**	**CCC-**	**D**
AAA	0	0	0	0	0	0	0
AA	0	0.015499	0	0.008795	0	0.001018	0
AA-	0	0.007653	0	0.016357	0	0.009222	0
A	0	0.004798	0	0.001643	0	0	0.00179
A-	0	0.009357	0	0.003793	0	0	0.001266
BBB	0	0.055705	0	0.007023	0	0.001336	0.003563
BBB-	0	0.093151	0	0.049235	0	0.011567	0.005211
BB	0.804949	0	0	0.123423	0	0.012818	0.016418
BB-	0	0.789015	0	0.145932	0	0.011718	0.021173
B	0.029954	0	0.892309	0	0	0.007011	0.05562
B-	0.028292	0	0	0.872399	0	0.048835	0.03796
CCC	0.013332	0	0.173902	0	0.56689	0	0.182397
CCC-	0.016184	0	0.169612	0	0	0.741203	0.056858
D	0	0	0	0	0	0	1

Table 19.24. *Probabilities of remaining in state i from years 2 to 7 and after to go in j-**II***

Tables 19.25, 19.26, 19.27 and 19.28 report $\phi_{ij}(s,t)$ (the element of the evolution equation matrix).

	AAA	**AA**	**AA-**	**A**	**A-**	**BBB**	**BBB-**
AAA	0.973331	3.23E-05	0.019182	1.87E-06	0.006311	2.30E-08	0.001037
AA	0.005264	0.964317	7.93E-05	8.58E-06	0.02358	6.34E-06	0.001527
AA-	0.005071	0.000101	0.962186	2.13E-05	0.01918	9.75E-06	0.004675
A	0.000823	0.014209	1.02E-05	0.976499	6.96E-05	7.74E-07	0.006184
A-	0.00111	0.009958	1.95E-05	5.49E-05	0.95434	2.97E-06	0.023357
BBB	0.000997	0.002635	1.08E-05	0.014953	3.16E-05	0.968121	6.51E-05
BBB-	0.000774	0.001457	1.13E-05	0.009248	1.43E-05	4.19E-05	0.946116
BB	6.72E-06	0.001589	1.39E-08	0.001036	1.66E-05	0.018562	6.41E-06
BB-	7.57E-06	0.002329	3.96E-08	0.002624	1.84E-05	0.012877	1.75E-05
B	4.30E-06	0.000761	4.08E-08	0.003514	4.35E-06	0.002204	2.02E-05
B-	4.39E-06	0.000435	3.27E-08	0.002285	5.53E-06	0.001897	1.72E-05
CCC	0.004433	3.51E-05	4.9E-05	0.004916	3.86E-06	0.005539	2.86E-05
CCC-	0.000115	2.04E-05	1.24E-06	0.00339	2.59E-07	0.002993	1.85E-05
D	0	0	0	0	0	0	0

Table 19.25. *Probabilities of being in j at time 3 given that at time 0 was in i-**I***

	BB	**BB-**	**B**	**B-**	**CCC**	**CCC-**	**D**
AAA	8.87E-08	3.78E-05	1.10E-08	4.01E-05	0	7.71E-06	1.91E-05
AA	9.85E-06	0.003022	4.00E-06	0.001399	0	0.00074	4.21E-05
AA-	1.1E-05	0.003049	2.19E-05	0.002839	0	0.002671	0.000164
A	3.58E-06	0.000493	4.60E-08	0.000766	0	1.33E-05	0.000929
A-	6.41E-06	0.001648	1.10E-07	0.00124	0	2.64E-05	0.008237
BBB	2.36E-06	0.010917	6.47E-06	0.00062	0	0.000719	0.00092
BBB-	8.89E-06	0.034942	1.9E-05	0.002013	0	0.002519	0.002836
BB	0.888772	0.000174	3.50E-06	0.07717	0	0.000701	0.011961
BB-	0.000454	0.853982	2.88E-05	0.106066	0	0.007009	0.014587
B	0.015196	1.5E-05	0.961312	7.42E-05	0	0.006078	0.010816
B-	0.021377	1.74E-05	0.000144	0.951864	0	0.015383	0.00657
CCC	0.003701	4.63E-05	0.039577	2.31E-05	0.924516	0.000114	0.017018
CCC-	0.00324	2.86E-05	0.016846	1.08E-05	0	0.902156	0.071179
D	0	0	0	0	0	0	1

Table 19.26. *Probabilities of being in j at time 3 given that at time 0 was in i-**II***

	AAA	**AA**	**AA-**	**A**	**A-**	**BBB**	**BBB-**
AAA	0.720751	0.000567	0.219784	0.000791	0.041623	0.00018	0.010661
AA	0.024902	0.817118	0.001019	0.00115	0.114982	0.00059	0.017173
AA-	0.021196	0.003109	0.724453	0.001831	0.168534	0.001272	0.030429
A	0.00389	0.040986	0.00013	0.820117	0.003373	0.001402	0.091656
A-	0.004086	0.039947	0.000146	0.008838	0.719879	0.002352	0.147647
BBB	0.004809	0.024665	0.000241	0.171344	0.00158	0.606366	0.012126
BBB-	0.003751	0.011209	0.000154	0.148196	0.000727	0.011129	0.540028
BB	0.000229	0.003772	7.32E-06	0.029637	0.000174	0.158403	0.001726
BB-	0.000156	0.003413	4.02E-06	0.021329	0.000162	0.107219	0.001524
B	0.000107	0.008409	2.46E-06	0.019755	0.000388	0.022386	0.001531
B-	0.000156	0.004512	3.57E-06	0.017595	0.000365	0.020248	0.001568
CCC	0.013874	0.002387	0.000357	0.023905	0.000151	0.028082	0.001715
CCC-	0.000702	0.002271	2.23E-05	0.019848	9.75E-05	0.022471	0.001308
D	0	0	0	0	0	0	0

Table 19.27. *Probabilities of being in j at time 10 given that at time 3 was in i-**I***

	BB	**BB-**	**B**	**B-**	**CCC**	**CCC-**	**D**
AAA	0.000752	0.002525	1.92E-05	0.001931	0	0.000526	0.000567
AA	0.000605	0.006092	0.000229	0.010242	0	0.002915	0.002983
AA-	0.001153	0.01247	0.00063	0.021024	0	0.007958	0.005942
A	0.00039	0.022088	4.57E-05	0.010469	0	0.002099	0.003356
A-	0.000883	0.037623	9.19E-05	0.023517	0	0.00391	0.011079
BBB	0.001973	0.091162	0.000496	0.0492	0	0.013299	0.02274
BBB-	0.005375	0.114259	0.001431	0.092938	0	0.027454	0.043349
BB	0.566792	0.008926	0.002051	0.136216	0	0.038457	0.05361
BB-	0.008583	0.535043	0.002757	0.188448	0	0.054521	0.076843
B	0.079673	0.001372	0.681411	0.009225	0	0.051393	0.124346
B-	0.064503	0.001347	0.005869	0.635897	0	0.092434	0.155501
CCC	0.035086	0.001532	0.21134	0.003613	0.341717	0.012572	0.323669
CCC-	0.034429	0.001224	0.183819	0.00365	0	0.386965	0.343194
D	0	0	0	0	0	0	1

Table 19.28. *Probabilities of being in j at time 10 given that at time 3 was in i-**II***

For example, 0.000493 represents the probability of being in state **BB-** at time 3 given that the rating evaluation was **A** at time 0.

Finally, in Tables 19.29 and 19.30 the $A_i(s,t)$ probabilities are reported. These elements give the probability that a firm, that is, at a given rating at time *s*, will not have a default up to the time *t*.

$A_i(s,t)$								
years		**AAA**	**AA**	**AA-**	**A**	**A-**	**BBB**	**BBB-**
0	1	1	1	1	0.999968	0.996925	0.999764	0.998961
0	2	0.999991	0.999987	0.999936	0.999423	0.993572	0.999435	0.998299
0	3	0.999981	0.999958	0.999836	0.999071	0.991763	0.99908	0.997164
0	4	0.999968	0.999911	0.999709	0.998793	0.988904	0.998726	0.995916
0	5	0.999956	0.999812	0.999476	0.998475	0.985322	0.998057	0.99488
0	6	0.999933	0.999633	0.999143	0.997611	0.982541	0.997147	0.992688
0	7	0.999894	0.999401	0.998674	0.996742	0.979183	0.99623	0.990739
0	8	0.999835	0.998948	0.997753	0.995935	0.976017	0.994732	0.986505
0	9	0.999749	0.998206	0.996333	0.994831	0.973398	0.992288	0.982006
0	10	0.999314	0.995626	0.991587	0.990844	0.964015	0.981222	0.957506
6	7	1	1	1	1	1	0.998896	0.995888
6	8	0.999993	0.999954	0.999886	0.99988	0.999831	0.997163	0.994382
6	9	0.999983	0.999888	0.999719	0.999635	0.999343	0.993383	0.991843
6	10	0.999846	0.999329	0.998343	0.997602	0.99556	0.981921	0.9661

Table 19.29. *Probability of not going into default from years s to t*

$A_i(s,t)$								
years		**BB**	**BB-**	**B**	**B-**	**CCC**	**CCC-**	**D**
0	1	0.996257	0.994695	0.996146	0.999959	0.996284	0.972282	0
0	2	0.992331	0.991607	0.993892	0.996661	0.991224	0.953744	0
0	3	0.988039	0.985413	0.989184	0.99343	0.982982	0.928821	0
0	4	0.984554	0.977635	0.983544	0.986963	0.976653	0.917664	0
0	5	0.976896	0.969593	0.976542	0.976842	0.96483	0.878435	0
0	6	0.966982	0.959714	0.969033	0.968583	0.950608	0.84797	0
0	7	0.959384	0.955638	0.95667	0.958528	0.937731	0.819395	0
0	8	0.950314	0.939394	0.947369	0.947336	0.918583	0.775739	0
0	9	0.938124	0.92374	0.929473	0.917325	0.819802	0.724016	0
0	10	0.897052	0.873723	0.876134	0.844328	0.700216	0.581562	0
6	7	0.99906	0.996682	0.987736	0.998923	0.960108	0.972323	0
6	8	0.996519	0.987823	0.977231	0.990127	0.917491	0.86902	0
6	9	0.989132	0.970269	0.960058	0.93397	0.798357	0.819678	0
6	10	0.961048	0.917886	0.892609	0.861352	0.640223	0.670431	0

Table 19.30. *Probability not going into default from* years *s to t*

19.8.3. *Non-homogenous downward backward example*

In this example, we use the same inputs as in the previous section and thus we will only report the results connected with the backward case.

In Tables 19.31 and 19.32, the probabilities $D_{ii}(u,s,t)$ of remaining in the state from *s* to *t* without any transition given that the system arrived at time *u* in state *i* and remained in this state from *u* to *s* are reported (backward recurrence time *s*-*u*).

Probabilities of no transition									
u	s	t	**AAA**	**AA**	**AA-**	**A**	**A-**	**BBB**	**BBB-**
0	0	1	0.872181	0.803709	0.86605	0.884755	0.859781	0.950899	0.925139
0	0	2	0.77428	0.740025	0.84935	0.81325	0.807756	0.92007	0.774197
0	0	3	0.699214	0.719164	0.68683	0.662296	0.792684	0.840488	0.721149
0	0	4	0.579474	0.607234	0.604108	0.562696	0.748248	0.677993	0.712246
0	0	5	0.496353	0.431912	0.44989	0.503661	0.676747	0.644425	0.652798
0	0	6	0.375035	0.370249	0.33209	0.466088	0.624805	0.594064	0.477243
0	0	7	0.302027	0.324532	0.266083	0.338748	0.505539	0.411709	0.314356
0	0	8	0.206926	0.270204	0.179743	0.223977	0.354685	0.229692	0.292188
0	0	9	0.12002	0.204777	0.139748	0.116117	0.17136	0.115246	0.121753
0	0	10	0.048179	0.031125	0.056915	0.081423	0.036674	0.083032	0.07514
2	6	7	0.89922	0.813412	0.754061	0.806292	0.768666	0.805227	0.665298
2	6	8	0.746307	0.542539	0.450393	0.56255	0.434755	0.682026	0.560027
2	6	9	0.216419	0.44434	0.315451	0.281127	0.208424	0.585145	0.328494
2	6	10	0.179681	0.022597	0.047193	0.049291	0.08731	0.02137	0.077891
4	6	7	0.917817	0.770902	0.939876	0.677056	0.604429	0.729913	0.823655
4	6	8	0.619728	0.552357	0.603088	0.386159	0.388559	0.620946	0.55475
4	6	9	0.285654	0.343981	0.476964	0.26715	0.251663	0.557772	0.283716
4	6	10	0.063009	0.042074	0.042642	0.129126	0.140237	0.041711	0.196981

Table 19.31. *Probabilities* $D_{ii}(u,s,t)$ - ***I***

Probabilities of no transition								
u	s	t	**BB**	**BB-**	**B**	**B-**	**CCC**	**CCC-**
0	0	1	0.985533	0.894593	0.83837	0.914242	0.883844	0.955493
0	0	2	0.947469	0.848314	0.693836	0.743081	0.725237	0.840868
0	0	3	0.858208	0.729037	0.618079	0.713725	0.682691	0.684999
0	0	4	0.701267	0.636971	0.575733	0.681344	0.578327	0.56305
0	0	5	0.577156	0.585617	0.537296	0.660163	0.502152	0.521922
0	0	6	0.463875	0.471031	0.401942	0.492227	0.480255	0.439965
0	0	7	0.359584	0.351512	0.302279	0.454226	0.344989	0.288244
0	0	8	0.242471	0.212772	0.241288	0.292127	0.193908	0.219999
0	0	9	0.136879	0.152736	0.185382	0.114899	0.14218	0.108098
0	0	10	0.064375	0.043551	0.011903	0.084761	0.043851	0.088229
2	6	7	0.729997	0.925012	0.937741	0.920945	0.925065	0.716396
2	6	8	0.396545	0.695417	0.558071	0.832468	0.672505	0.553408
2	6	9	0.221682	0.459819	0.262421	0.396267	0.342161	0.283633
2	6	10	0.129046	0.123599	0.110453	0.036925	0.02891	0.051503
4	6	7	0.831486	0.872701	0.785129	0.91658	0.664807	0.865949
4	6	8	0.68755	0.552467	0.501403	0.567741	0.428559	0.583168
4	6	9	0.469288	0.303929	0.426686	0.248322	0.295415	0.321166
4	6	10	0.040333	0.029382	0.049693	0.201695	0.163986	0.150543

Table 19.32. *Probabilities* $D_{ii}(u,s,t)$ - ***II***

	AAA	**AA**	**AA-**	**A**	**A-**	**BBB**	**BBB-**
AAA	0.87914	7.43E-05	0.109719	7.32E-05	0.006752	5.27E-06	0.003312
AA	0.006296	0.934405	5E-05	0.000182	0.02957	0.000291	0.008368
AA-	0.004272	0.000385	0.909481	0.00054	0.045075	0.000118	0.023819
A	0.002963	0.041231	6.75E-05	0.912063	0.000969	0.000102	0.030523
A-	0.004478	0.035515	7.56E-05	0.00142	0.871933	0.000163	0.063676
BBB	0.002678	0.00514	5.46E-05	0.058214	0.000107	0.85594	0.001179
BBB-	0.001799	0.003705	3.45E-05	0.024706	0.0001	0.001665	0.826728
BB	6.51E-05	0.003005	3.21E-07	0.011852	8E-05	0.041455	0.000366
BB-	4.31E-05	0.003697	2.71E-07	0.004487	8.75E-05	0.024516	0.000136
B	2.56E-05	0.002082	1.07E-07	0.00639	3.82E-05	0.006298	0.000109
B-	3.35E-05	0.002076	1.67E-07	0.005122	5.98E-05	0.004665	9.97E-05
CCC	0.012057	0.000273	0.000163	0.014286	2.06E-05	0.010276	0.000285
CCC-	0.000297	0.000207	3.61E-06	0.010776	3.04E-06	0.01226	0.000238
D	0	0	0	0	0	0	0

Table 19.33. *Probabilities* $\phi_{ij}(2,4,8)$ - ***I***

	BB	**BB-**	**B**	**B-**	**CCC**	**CCC-**	**D**
AAA	5.57E-07	0.000554	2.00E-07	0.000324	0	1.67E-05	2.95E-05
AA	6.04E-05	0.01208	7.3E-05	0.00659	0	0.00156	0.000475
AA-	4.05E-05	0.005448	0.000129	0.005271	0	0.004953	0.000469
A	1.8E-05	0.006937	4.60E-06	0.003268	0	0.000261	0.001594
A-	4.17E-05	0.010847	1.07E-05	0.008864	0	0.000479	0.002497
BBB	9.45E-05	0.05323	0.000153	0.01481	0	0.004568	0.003831
BBB-	0.000542	0.06678	0.000358	0.055484	0	0.010978	0.00712
BB	0.807165	0.000809	0.000606	0.102833	0	0.020182	0.011582
BB-	0.000537	0.816197	0.000448	0.111314	0	0.017023	0.021514
B	0.029385	0.000196	0.891416	0.000712	0	0.017088	0.046261
B-	0.013616	0.000117	0.00294	0.83323	0	0.065388	0.072652
CCC	0.013842	0.000186	0.10049	0.000363	0.723444	0.003141	0.121172
CCC-	0.013936	0.000276	0.09612	0.000494	0	0.75511	0.110281
D	0	0	0	0	0	0	1

Table 19.34. *Probabilities* $\phi_{ij}(2,4,8)$ - ***II***

In Tables 19.33, 19.34, 19.35 and 19.36, some of the evolution equation sub-matrices of the $\phi_{ij}(u,s,t)$ are reported.

For example, 0.006937 represents the probability of being in state **BB-** at time 8 given that the rating evaluation was **A** at time 4 and the system entered into this state at time 2 (backward recurrence time 4-2).

	AAA	AA	AA-	A	A-	BBB	BBB-
AAA	0.893323	0.000122	0.08903	9.24E-05	0.013534	4.33E-06	0.003405
AA	0.006692	0.886695	0.000113	0.000456	0.074917	0.00097	0.008796
AA-	0.004456	0.001082	0.821541	0.001036	0.105953	0.000486	0.031362
A	0.007464	0.043065	0.000116	0.858171	0.003494	0.000777	0.06479
A-	0.004565	0.051651	0.000171	0.002344	0.771791	0.001026	0.132614
BBB	0.002391	0.013464	9.49E-05	0.057016	0.00083	0.854738	0.003984
BBB-	0.002063	0.009745	5.01E-05	0.048088	0.000341	0.004701	0.70638
BB	3.04E-05	0.005834	1.25E-07	0.008943	0.000311	0.049101	0.000341
BB-	3.76E-05	0.003823	2.20E-07	0.014951	0.000228	0.059871	0.000431
B	2.56E-05	0.004185	1.90E-07	0.007884	0.00018	0.010822	0.000403
B-	2.74E-05	0.00255	1.73E-07	0.006393	5.64E-05	0.010861	0.000339
CCC	0.026797	0.00159	0.000761	0.026286	0.000126	0.023591	0.001191
CCC-	0.000717	0.001173	1.76E-05	0.012282	2.26E-05	0.013133	0.000242
D	0	0	0	0	0	0	0

Table 19.35. *Probabilities* $\phi_{ij}(5,7,10)$ - ***I***

	BB	BB-	B	B-	CCC	CCC-	D
AAA	2.15E-07	0.00033	5.33E-08	8.77E-05	0	1.81E-05	5.31E-05
AA	0.000278	0.008518	1.39E-05	0.009602	0	0.001855	0.001094
AA-	0.000531	0.009151	0.000274	0.013886	0	0.005551	0.004692
A	4.03E-05	0.01449	3.74E-06	0.004256	0	0.000759	0.002575
A-	0.000128	0.021767	7.39E-06	0.009632	0	0.001292	0.003012
BBB	0.000347	0.046795	6.12E-05	0.010801	0	0.003264	0.006214
BBB-	0.000414	0.123791	0.000338	0.063061	0	0.014662	0.026367
BB	0.74951	0.001672	0.000397	0.126385	0	0.021406	0.03607
BB-	0.006158	0.636701	0.000714	0.183819	0	0.041211	0.052055
B	0.048595	0.000295	0.808251	0.003852	0	0.029791	0.085716
B-	0.055889	0.000264	0.001062	0.734109	0	0.09917	0.089279
CCC	0.019177	0.000997	0.164981	0.001588	0.43202	0.005078	0.295819
CCC-	0.029484	0.00043	0.161747	0.001591	0	0.536183	0.242977
D	0	0	0	0	0	0	1

Table 19.36. *Probabilities* $\phi_{ij}(5,7,10)$ - ***II***

$A_i(s,t)$									
u	*s*	*t*	**AAA**	**AA**	**AA-**	**A**	**A-**	**BBB**	**BBB-**
0	0	1	1	1	1	0.999968	0.996925	0.999764	0.998961
0	0	2	0.999991	0.999987	0.999936	0.999423	0.993572	0.999435	0.998299
0	0	3	0.999981	0.999958	0.999836	0.999071	0.991763	0.99908	0.997164
0	0	4	0.999968	0.999911	0.999709	0.998793	0.988904	0.998726	0.995916
0	0	5	0.999956	0.999812	0.999476	0.998475	0.985322	0.998057	0.99488
0	0	6	0.999933	0.999633	0.999143	0.997611	0.982541	0.997147	0.992688
0	0	7	0.999894	0.999401	0.998674	0.996742	0.979183	0.99623	0.990739
0	0	8	0.999835	0.998948	0.997753	0.995935	0.976017	0.994732	0.986505
0	0	9	0.999749	0.998206	0.996333	0.994831	0.973398	0.992288	0.982006
0	0	10	0.999314	0.995626	0.991587	0.990844	0.964015	0.981222	0.957506
2	6	7	1	1	1	0.999717	0.99652	0.998982	0.999799
2	6	8	0.999993	0.999909	0.999681	0.998935	0.991975	0.997865	0.995446
2	6	9	0.999986	0.999532	0.998959	0.998718	0.989737	0.995587	0.990746
2	6	10	0.999891	0.997734	0.995524	0.996719	0.984762	0.985371	0.969673
4	6	7	1	1	1	0.999957	0.999766	0.999656	0.998659
4	6	8	0.999996	0.999844	0.999806	0.99937	0.999451	0.998292	0.995586
4	6	9	0.99999	0.999614	0.998885	0.998971	0.999054	0.996909	0.991069
4	6	10	0.999889	0.998192	0.994393	0.997174	0.996021	0.99036	0.966736

Table 19.37. *Probability of not defaulting from s to t with backward recurrence time s-u-**I***

Finally, in Tables 19.37 and 19.38 the probabilities of never going into default are reported. These elements give the probability that a firm, that is, at a given rating at time *s*, will not have a default up to time *t*, given that it had the rating at time *u* (backward recurrence time *s*-*u*).

$A_i(s,t)$								
u	*s*	*t*	**BB**	**BB-**	**B**	**B-**	**CCC**	**CCC-**
0	0	1	0.996257	0.994695	0.996146	0.999959	0.996284	0.972282
0	0	2	0.992331	0.991607	0.993892	0.996661	0.991224	0.953744
0	0	3	0.988039	0.985413	0.989184	0.99343	0.982982	0.928821
0	0	4	0.984554	0.977635	0.983544	0.986963	0.976653	0.917664
0	0	5	0.976896	0.969593	0.976542	0.976842	0.96483	0.878435
0	0	6	0.966982	0.959714	0.969033	0.968583	0.950608	0.84797
0	0	7	0.959384	0.955638	0.95667	0.958528	0.937731	0.819395
0	0	8	0.950314	0.939394	0.947369	0.947336	0.918583	0.775739
0	0	9	0.938124	0.92374	0.929473	0.917325	0.819802	0.724016
0	0	10	0.897052	0.873723	0.876134	0.844328	0.700216	0.581562
2	6	7	0.998572	0.978718	0.969794	0.977368	0.987032	0.979671
2	6	8	0.996016	0.970403	0.954563	0.96299	0.97334	0.95212
2	6	9	0.99033	0.962454	0.940606	0.946127	0.880292	0.891065
2	6	10	0.956603	0.928904	0.89047	0.86583	0.756136	0.732838
4	6	7	0.999171	0.998713	0.992329	0.984648	0.99576	0.963201
4	6	8	0.992429	0.986815	0.980369	0.971697	0.955405	0.942493
4	6	9	0.986339	0.980006	0.956142	0.947635	0.820178	0.88657
4	6	10	0.958009	0.942448	0.905999	0.888396	0.684876	0.725764

Table 19.38. *Probability of not defaulting from s to t with backward recurrence time s-u-**II***